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AUTOMORPHISMS OF 2-(22, 8, 4) DESIGNS

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Dedicated to Professor Haim Hanani on the occasion of his 75th birthday.

It is shown that a 2-(22, 8, 4) design cannot possess any nontrivial automorphisms of an odd order.

1. Introduction

The smallest, with respect to the number of points or blocks, parameter set for a balanced incomplete block design, i.e. a 2-(v, k, λ) design, for which the existence question is still unsolved, is 2-(22, 8, 4), i.e. $v = 22$, $b = 33$, $r = 12$, $k = 8$, $\lambda = 4$. This is the smallest case left open in Table 5.23 of the remarkable Hanani's article [7]. Many of the open problems from that table have been resolved during the last decade, some of then by Professor Hanani himself (cf. Mathon and Rosa [11]). However, the existence of the smallest and most challenging 2-(22, 8, 4) design is still in doubt.

In this paper we investigate possible automorphism groups of a design with such parameters and show that if one exists, its full automorphism group must be either a 2-group, or trivial. Our method is based on examination of possible orbit structures of cyclic automorphism groups of a prime order by use of tactical decompositions.

An essential case of automorphisms of order 3 fixing exactly one point has been recently investigated by Kapralov [9], who found all (exactly 53) possible orbit structures and showed (partially by computer) that none of those yields a design. We show in this paper that for an odd prime order automorphism of any other type, there is no possible orbit structure at all. Our proof does not involve any computer computations.

2. Preliminaries

We asssume that the reader is familiar with the basic notions and facts from design theory (cf. e.g. [3, 4, 5, 8, 13]).

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As usual, the total number of blocks in a 2-(v, k, λ) design is denoted by b, and the number of blocks containing a given point – by r.

The following easily checked statement is a variation of a similar but stronger result for symmetric 2-designs (cf. [1]).

Lemma 2.1. *If p is a prime being an order of an automorphism of a* 2-(v, k, λ) *design with $v > k$, then either p divides v or $p \leqslant r$.*

Applied for the parameters 2-(22, 8, 4), this gives as a corollary the following

Lemma 2.2. *The only primes which might be orders of automorphisms of a* 2-(22, 8, 4) *design, are* 2, 3, 5, 7 *or* 11.

The next result is a special case of Theorem 1.46 from [8] (see also [3, Th. 4, 17]).

Lemma 2.3. *If v' (resp. b') is the number of point (resp. block) orbits of a nontrivial* 2-(v, k, λ) *design with respect to a given automorphism group, then*

$$0 \leqslant b' - v' \leqslant b - v.$$

In the sequal we shall use frequently the following result due to Hamada and Kobayashi [6]:

Lemma 2.4. *Any two blocks in a* 2-(22, 8, 4) *design can have at most* 4 *common points. More precisely, if n_i denotes the number of blocks intersecting a given block in exactly i points, then there are* 4 *possible types of blocks according to their block intersection numbers* (*Table* 1).

Given a design D with an automorphism group G, the orbit matrix $M = (m_{ij})$ of D with respect to G is defined as a matrix whose rows and columns are indexed by the point and block orbits of D under G respectively, where m_{ij} is the number of points from the ith point orbit contained in a block from the jth block orbit. In other words, M is a matrix corresponding to the tactical decomposition of D defined by the action of G.

Let r_j (resp. k_i) denote the length of the jth block (resp. ith point) orbit, and let

Table 1. Block intersection numbers of a 2-(22, 8, 4) design.

Type	n_0	n_1	n_2	n_3	n_4
1	0	0	12	16	4
2	0	1	9	19	3
3	0	2	6	22	2
4	1	0	6	24	1

b' (resp. v') be the total number of block orbits. In this notation, the orbit matrix M satisfies the following equations:

$$\sum_{j=1}^{b'} r_j m_{ij} = k_i r, \quad 1 \leq i \leq v', \tag{2.1}$$

$$\sum_{j=1}^{b'} r_j m_{ij}(m_{ij} - 1) = k_i(k_i - 1)\lambda, \quad 1 \leq i \leq v', \tag{2.2}$$

$$\sum_{j=1}^{b'} r_j m_{cj} m_{dj} = k_c k_d \lambda \quad \text{for } c \neq d. \tag{2.3}$$

If G is a cyclic group of a prime order p then any orbit length is either p or 1. In particular, considering a nontrivial (i.e. of length p) point orbit and denoting by $s = s_i$ the number of blocks fixed by G and containing all points from that (ith) orbit, equations (2.1)–(2.3) reduce to the following:

$$\sum_{j:r_j=p} m_{ij} = r - s_i, \tag{2.4}$$

$$\sum_{j:r_j=p} m_{ij}(m_{ij} - 1) = (p-1)(\lambda - s_i), \tag{2.5}$$

$$\sum_{j:r_j=p} m_{cj} m_{dj} = p(\lambda - s_{cd}), \quad (c \neq d), \tag{2.6}$$

where s_{cd} denotes the number of fixed blocks containing the cth and dth point orbit. Combined with (2.4), (2.5) gives also

$$\sum_{j:r_j=p} m_{ij}^2 = p(\lambda - s_i) + r - \lambda. \tag{2.7}$$

An evident necessary condition for the existence of a design with a given automorphism group is the existence of an integral matrix $M = (m_{ij})$ satisfying the above system of equations.

3. Automorphisms of order 11

According to Lemma 2.2, the largest prime which can possibly be an order of an automorphism of a 2-(22, 8, 4) design, is 11.

The impossibility of an automorphism without fixed points has been mentioned by Baartmans and Danhof [2]: the system of Equations (2.4)–(2.6) then has no solution.

Suppose f is an automorphism of order 11 fixing 11 points. Then by Lemma 2.3 f must fix at least 11 blocks. Any two blocks fixed by f must consist entirely of points fixed by f and hence they have at least 5 common points, a contradiction to Lemma 2.4.

4. Automorphisms of order 7

Since $b = 33 \equiv 5 \pmod 7$, an automorphism of order 7 must fix at least 5 blocks. Since a point orbit of length 7 can be contained in at most one fixed block (by Lemma 2.4), this rules out immediately an automorphism fixing 1 or 8 points. If there are 15 fixed points then by Lemma 2.3 there have to be at most 2 blocks orbits of length 7. However, the corresponding system (2.4)–(2.7) has no solution for $p = 7$ and $s_i < 2$.

5. Automorphisms of order 5

Since $b = 33 \equiv 3 \pmod 5$, there must be at least 3 fixed blocks. According to Lemma 2.4, a point orbit of length 5 can be contained in at most one fixed block. The only (up to permutation) solutions of (2.4)–(2.7) for $p = 5$ and $s_i < 2$ are $(1, 1, 2, 2, 3, 3)$ $(s_i = 0)$ and $(1, 1, 2, 2, 2, 3)$ $(s_i = 1)$. Therefore, there are 3 fixed blocks, whence by Lemma 2.3 there are only 2 fixed points. However, a fixed block must contain at least 3 fixed points, a contradiction.

6. Automorphisms of order 3

The following lemma gives an upper bound for the number of blocks fixed by an automorphism of order 3.

Lemma 6.1. *An automorphism of order* 3 *of a* 2-(v, k, λ) *design can fix at most* $b - 3r + 3\lambda$ *blocks.*

Proof. Let S be a point orbit of length 3 and let n_i be the number of blocks containing exactly i points from S. Evidently

$$\begin{aligned} n_0 + n_1 + n_2 + n_3 &= b, \\ n_1 + 2n_2 + 3n_3 &= 3r, \\ n_2 + 3n_3 &= 3\lambda. \end{aligned}$$

Since each fixed block contains either 3 or none points from S, the total number of fixed blocks does not exceed

$$n_0 + n_3 = b - 3(r - \lambda). \quad \square$$

Corollary 6.2. *An automorphism of order* 3 *of a* 2-(22, 8, 4) *design fixes at most* 9 *blocks.*

Lemma 6.3. *Given a* 2-(22, 8, 4) *design D with an automorphism f of order* 3, *and* *a block B not fixed by f, there are at least 4 point orbits of length 3 intersecting B in either 1 or 2 points.*

Proof. Let B be a block not fixed by f. Denote by t the number of points fixed by f and contained in B, and let m_i $(i = 1, 2, 3)$ denote the number of point orbits of length 3 intersecting B in exactly i points. Evidently

$$t + m_1 + 2m_2 + 3m_3 = 8. \tag{6.1}$$

On the other hand,

$$|B \cap Bf| = t + m_2 + 3m_3 \leqslant 4,$$

whence

$$m_1 + m_2 \geqslant 4.$$

In particular, there are at least 4 point orbits of length 3. □

Corollary 6.4. *An automorphism of order 3 of a 2-(22, 8, 4) design fixes at most 10 points.*

As we have already mentioned, the nonexistence of a 2-(22, 8, 4) design with an automorphism of order 3 fixed exactly 1 point has been proved by Kapralov [9]. Thus we have to consider automorphisms fixing 4, 7 or 10 points.

Lemma 6.5. *If an automorphism of order 3 of a 2-(22, 8, 4) design fixes more than 1 point then each fixed point is contained in at least 3 fixed blocks.*

Proof. Since $r = 12 \equiv 0 \pmod 3$, the number of fixed blocks through a fixed point is a multiple of 3. Any pair of fixed points is contained in $4 \equiv 1 \pmod 3$ blocks, hence one or all of these 4 blocks must be fixed. Thus each fixed point occurs in a fixed block, and consequently, in at least 3 fixed blocks. □

Suppose that D is a 2-(22, 8, 4) design with an automorphism f of order 3. The orbit matrix M with respect to the cyclic group generated by f can be presented in the following form

$$M = \begin{vmatrix} T & U \\ V & W \end{vmatrix}, \tag{6.3}$$

where $T = (t_{ij})$ has rows and columns indexed by the fixed points and blocks; $U = (u_{ij})$ has rows indexed by fixed points and columns indexed by nontrivial block orbits; $V = (v_{ij})$ has rows indexed by nontrivial point orbits and columns by fixed blocks; and $W = (w_{ij})$ has rows and columns indexed by nontrivial point and block orbits.

7. Automorphisms of order 3 fixing 10 points

In this case there are exactly 4 point orbits of length 3, i.e. the matrix (V, W) from (6.3) has exactly 4 rows. By Lemma 6.3 each entry of W is either 1 or 2.

Suppose that there are x fixed blocks, and hence $y=(33-x)/3$ blocks orbits of length 3. Let $(v_{i1},\ldots,v_{ix},w_{i1},\ldots,w_{iy})$ be a row of (V, W), and denote by q_j (resp. p_j) the number of entries among $v_{i1},\ldots,v_{ix}$ (resp. $w_{i1},\ldots,w_{iy}$) equal to j $(0\leqslant j\leqslant 3)$. Clearly

$$\begin{aligned} q_3+2p_2+p_1&=12,\\ q_3+\ p_2\qquad &=4,\\ p_2+p_1&=y, \end{aligned}$$

whence $y=8$, and $x=9$, i.e. there are exactly 9 fixed blocks.

Equations (2.4)–(2.7) now give the following possibilities for the rows of (V, W) (Table 2):

Table 2. Rows of (V, W).

Type	V									W							
i	0	0	0	0	0	0	0	0	0	2	2	2	2	1	1	1	1
ii	3	0	0	0	0	0	0	0	0	2	2	2	1	1	1	1	1
iii	3	3	0	0	0	0	0	0	0	2	2	1	1	1	1	1	1
iv	3	3	3	0	0	0	0	0	0	2	1	1	1	1	1	1	1
v	3	3	3	3	0	0	0	0	0	1	1	1	1	1	1	1	1

By equation (2.6) and Lemma 2.4 the scalar product of pair of rows of W must be either 9 or 12. This is possible only for pairs of rows of the following types: (i, v), (ii, iv), (iii, iii), (iii, iv), (iv, v). This excludes rows of type i or v. Furthermore, there is at most one row of type iv, and such a row can be combined with at most 2 rows of type iii; hence a row of type iv is also excluded. Eventually, up to permutation of rows and columns, (V, W) looks as follows:

$$(V,W)=\left|\begin{matrix} 3&3&0&0&0&0&0&0&0&2&2&1&1&1&1&1&1\\ 0&0&3&3&0&0&0&0&0&1&1&2&2&1&1&1&1\\ 0&0&0&0&3&3&0&0&0&1&1&1&1&2&2&1&1\\ 0&0&0&0&0&0&3&3&0&1&1&1&1&1&1&2&2 \end{matrix}\right|.$$

Hence there are 8 fixed blocks each containing 5 fixed points, one fixed block (say B) consisting entirely of fixed points, and each nonfixed block contains 3 fixed points. Let P be a fixed point belonging to B. Denote by R_1 the number of fixed blocks other than B and containing P, and let R_2 be the number of nonfixed blocks containing P. Counting in two ways the number of blocks containing P and another fixed point, one gets:

$$7 + 4R_1 + 2R_2 = 9 \cdot 4,$$

a contradiction.

Therefore, there is no design with an automorphism of order 3 fixing 10 points.

8. Automorphisms of order 3 fixing 7 points

The number of point orbits is now 12, hence by Lemma 2.3 and Corollary 6.2 there are 3, 6 or 9 fixed blocks.

Each fixed block contains 2 or 5 fixed points. By Lemma 6.5 each fixed point is contained in at least 3 fixed blocks. If there are only 3 fixed blocks then each of the 7 fixed points must belong to each of the 3 fixed blocks, which contradicts to Lemma 2.4. Hence there are 6 or 9 fixed blocks.

Assume that there are exactly 6 fixed blocks. Denote by n_2 (resp. n_5) the number of blocks containing exactly 2 (resp. 5) fixed points. Evidently

$$n_2 + n_5 = 6,$$

and since each fixed point is contained in at least 3 fixed blocks (Lemma 6.5), we have also

$$2n_2 + 5n_5 \geqslant 7 \cdot 3,$$

whence $n_5 \geqslant 3$.

Two fixed blocks, each containing 5 fixed points, must intersect in at least 3 fixed points. Each pair of such a triple of points is contained in at least 2, and hence in exactly 4 fixed blocks. Therefore, each point of such a triple occurs in at least 4 fixed blocks, hence by the proof of Lemma 6.5 in at least 6 fixed blocks, i.e. in all fixed blocks, which leads to a contradiction with $\lambda = 4$.

Therefore, there must be exactly 9 fixed blocks.

Proceeding as in the case of 10 fixed points (Section 7), it can be seen that the matrix (V, W) must consist of 5 rows of type iii (cf. Table 2). However, it is readily seen that the matrix (7.1) cannot be extended with a 5th row of type iii so that the scalar product of each pair of rows to be either 9 or 12.

9. Automorphisms of order 3 fixing 4 points

In this case a fixed block must consist of 2 fixed points and 2 point orbits of length 3. Each pair of fixed points is contained in 4 blocks, either one or all of them being fixed. However, if there is a pair of fixed points contained in 4 fixed blocks then some pair of these 4 blocks must have at least 5 common points, in conflict with Lemma 2.4. Thus each pair of fixed points is contained in precisely one fixed block, and hence there are exactly 6 fixed blocks.

In the notation of (6.3), the matrix T now is an incidence matrix of the trivial 2-(4, 2, 1) design, e.g.

$$T = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{vmatrix}. \tag{9.1}$$

Up to permutation of rows and columns there are 3 possibilities for the matrix U:

$$U = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix}. \tag{9.2}$$

$$U = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}. \tag{9.3}$$

$$U = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{vmatrix}. \tag{9.4}$$

Equations (2.4)–(2.7) give the following possibilities for rows of (V, W) (Table 3):

Lemma 9.1. *There is no design with a matrix U of the form* (9.2).

Proof. Assume that U has the form (9.2). Then by Lemma 6.3 each block from the only block orbit of length 3 containing 4 fixed points must contain at most one point from a point orbit of length 3. Thus the orbit matrix M has the following

Table 3. Rows of (V, W).

Type	V						W								
i	0	0	0	0	0	0	2	2	2	2	1	1	1	1	0
ii	3	0	0	0	0	0	2	2	2	1	1	1	1	1	0
iii	3	3	0	0	0	0	2	2	1	1	1	1	1	1	0
iv	0	0	0	0	0	0	2	1	1	1	1	1	1	1	3
v	3	3	3	0	0	0	2	1	1	1	1	1	1	1	0
vi	3	0	0	0	0	0	1	1	1	1	1	1	1	1	3
vii	3	3	3	3	0	0	1	1	1	1	1	1	1	1	0

form:

$$M=\left|\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & & & & & & 0 & & & & & & & & \\
 & & & & & & 0 & & & & & & & & \\
 & & & * & & & 1 & & & & * & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & &
\end{array}\right|. \tag{9.5}$$

Hence the first two rows of the submatrix W of (9.5) contain a common zero coordinate, and therefore, such a row cannot be of type iv or vi. Since the scalar product of two rows of W must be either 9 or 12, the first two rows can be of the following types; (i, vii), (ii, v), (iii, iii), (iii, v), (v, vii). The scalar product of a row of (V, W) after replacing each entry 3 in V by 1 with each row of (T, U) must be equal to 4. This is not possible if one of the first two rows of W is of type i, ii, iii, iv, or v. This completes the proof. □

In general, if $(t_{i1}, \ldots, t_{i6}, u_{i1}, \ldots, u_{i9})$, $1 \le i \le 4$ are the rows of (T, U), then any row $(v_1, \ldots, v_6, w_1, \ldots, w_9)$ of (V, W) must satisfy the following equations (cf. (2.6)):

$$\sum_{j=1}^{6} v_j t_{ij} + 3\sum_{j=1}^{9} w_j u_{ij} = 12, \quad i = 1, 2, 3, 4.$$

Any solution of (9.6) must be of type i–vii (Table 3).

Lemma 9.2. *If U is of the form* (9.3) *or* (9.4), *then there is no row of (V, W) of type* iv, vi, *or* vii.

Proof. Assume that U has the form (9.3). Then the system of Equations (9.6) looks as follows:

$$\begin{array}{l}
v_1 + v_2 + v_3 + 3w_1 + 3w_2 + 3w_3 = 12, \\
v_1 + v_4 + v_5 + 3w_1 + 3w_4 + 3w_5 = 12, \\
v_2 + v_4 + v_6 + 3w_1 + 3w_6 + 3w_7 = 12, \\
v_3 + v_5 + v_6 + 3w_2 + 3w_4 + 3w_6 = 12.
\end{array}$$

If some $w_i = 3$ then there should be some $w_j = 0$. Hence a solution of type iv or vi is not possible.

Assume now that there is a solution of type vii. Up to permutation, there are only two possibilities: $v_1 = \cdots = v_4 = 3$, $v_5 = v_6 = 0$; or $v_1 = v_6 = 0$, $v_2 = \cdots = v_5 = 3$ (cf. (9.1)). In the first case two of w_1, w_2, w_3 must be zero, a contradiction (see Table 3). In the second case, if $w_1 = 1$ then one of w_2 or w_3, as well as one of w_4 or w_5 must be zero, a contradiction; if $w_1 = 0$, then the first 3 equations imply $w_2 = \cdots = w_7 = 1$, whence the 4th equation is violated.

The case when U has the form (9.4) is treated similarly; the system of Equations (9.6) again does not admit any solution of type iv, vi or vii. □

Using the fact that the matrix V contains 12 entries equal to 3 and 24 zeros, Lemmas 9.1, 9.2 and Eq. (2.4–2.7) imply the following

Lemma 9.3. *There are* 6 *possibilities for the types of the rows of the matrix* (V, W):

$$1(\mathrm{i}) + 1(\mathrm{ii}) + 1(\mathrm{iii}) + 3(\mathrm{v}), \tag{9.7}$$

$$3(\mathrm{ii}) + 3(\mathrm{v}), \tag{9.8}$$

$$1(\mathrm{i}) + 3(\mathrm{iii}) + 2(\mathrm{v}), \tag{9.9}$$

$$2(\mathrm{ii}) + 2(\mathrm{iii}) + 2(\mathrm{v}), \tag{9.10}$$

$$1(\mathrm{ii}) + 4(\mathrm{iii}) + 1(\mathrm{v}), \tag{9.11}$$

$$6(\mathrm{iii}). \tag{9.12}$$

Here $a(b)$ means a rows of type b.

Let us now consider the incidence structure F with "points" the 6 nontrivial point orbits and "blocks" the 6 fixed blocks. Each block of F consists of a pair of points and (by Lemma 2.4) there are no repeated blocks. Hence F is a collection of 6 distinct 2-subsets of a given 6-set, or equivalently, F is a 6-subset of the set of all 15 2-subsets of the point set. The set of all such $\binom{15}{6}$ 6-subsets is divided into 21 orbits under the action of the symmetric group of degree 6 on the point set (cf. e.g. Kramer and Mesner [10]). Thus there are at most 21 possible configurations for F. By Lemmas 9.1 and 9.2 each point of F occurs in at most 3 blocks, which reduces the possibilities from 21 to 14.

Let us define a graph G with vertices the points of F and edges the blocks of F. By definition G has 6 vertices and 6 edges. Using Equations (2.4)–(2.7), the possible types of rows of (V, W) (Table 3), and Lemmas 6.3, 9.1, 9.2, 9.3, it can be seen that the graph G must possess the following properties:

9.4. Each vertex is of degree at most 3.

9.5. A vertex of degree 0, 1, 2 or 3 corresponds to a row of (V, W) of type i, ii, iii, or v respectively.

9.6. Two vertices of degree 3 are necessarily adjacent.

9.7. Any vertex of degree 1 is adjacent to a vertex of degree 3.

9.8. A vertex of degree 3 is adjacent to at most one vertex of degree 1.

9.9. A triple of vertices of degree 2 cannot form a complete graph of size 3.

9.10. Given a vertex P of degree 3, there is at most one vertex of degree 2 nonadjacent to P.

9.11. If G contains a pair of adjacent vertices of degree 1 and 3 respectively, then there is no vertex of degree 0 in G.

9.12. The scalar product of two rows of W corresponding to a pair of adjacent (resp. nonadjacent) vertices of G is 9 (resp. 12).

The properties 9.4–9.12 reduce the possible configurations for F to the following 4 ones:

$$F_1 = \{12, 13, 14, 23, 25, 45\},$$

$$F_2 = \{12, 16, 23, 34, 45, 56\},$$

$$F_3 = \{12, 14, 15, 23, 26, 34\},$$

$$F_4 = \{12, 13, 14, 23, 25, 36\}.$$

Using 9.12, it is straightforward to check that (up to permutation of rows and columns) a triple of rows of (V, W) of type iii corresponding to 3 vertices of G of degree 2, two adjacent and the third nonadjacent to any of them, looks as follows:

$$\begin{matrix} 3 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 1. \end{matrix} \tag{9.13}$$

The matrix (9.13) cannot be extended by a row of type i. This eliminates F_1.

Similarly, the matrix (9.13) cannot be extended by a row of type iii, having scalar product 12 with the first two rows and 9 with the third row. Thus F_2 is also impossible.

Up to permutation, there is only one possibility for a triple of rows of (V, W) of type v, iii, ii respectively, corresponding to a triple of pairwise nonadjacent vertices of G:

$$\begin{matrix} 3 & 3 & 3 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1. \end{matrix} \tag{9.14}$$

The matrix (9.14) cannot be extended by a row of type ii having scalar product 9 with the first row, and 12 with each of the remaining two rows of (9.14). This eliminates F_3.

Finally, there is exactly one (up to permutation) matrix (V, W) corresponding to F_4:

$$\begin{array}{ccccccccccccccc}
3 & 3 & 3 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
3 & 0 & 0 & 3 & 3 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 3 & 0 & 3 & 0 & 3 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 3 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 1 & 1.
\end{array} \tag{9.15}$$

The corresponding matrix U has to be of the form (9.4). However, the system (9.6) has only two solutions for a row of (T, U): 110100000000111 and 001011000111000. Hence, the matrix (9.15) is not extendable to an orbit matrix.

Consequently, there is no 2-(22, 8, 4) design with an automorphism of order 3 fixing exactly 4 points.

Combined with the Kapralov result [9], the above results can be summarized in the following.

Theorem 9.13. *The full automorphism group of a* 2-(22, 8, 4) *design must be either a* 2-*group, or trivial.*

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Final remark

The authors have been informed by one of the referees that an investigation of 2-(22, 8, 4) designs has been recently carried out by Hall, Roth, van Rees and Vanstone [12]. Since the last paper had not yet been published by the time of submission of our paper, we were unable to make any comparison with its results.

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